Dominant Strategy Implementation

Micro Theory II class notes, Temple University

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Outline

What Is It?

- We study the possibility of implementing a social choice correspondence (or function) in dominant strategies.
- We want to start with a social desideratum, embodied in a SCC or SCF and make up a mechanism that, when played in the actual environment of a realized economy, will give each agent a dominant strategy.
- When each agent plays that dominant strategy, a desired outcome (a member of the SCC) will be achieved.

Setup

Definition

Let the set of agents be $\mathscr{I} = \{1, 2, \ldots, I\}$. Let the set of possible alternatives be X. For each $i \in \mathscr{I}$, let Θ_i be a non-empty set. Let θ_i denote a type of agent $i \in \mathscr{I}$. An environment is a profile of types $\theta = (\theta_1, \ldots, \theta_I)$.

Setup, Continued

Definition

For each agent $i \in \mathscr{I}$, let $R^i(\theta_i)$ denote this agent's preference relation, that is, for any two alternatives $x, y \in X$, $x\, R^i(\theta_i)\, y$ denotes the statement "agent i, when of type θ_i, considers alternative x at least as good as alternative y."

Setup, Continued

Definition

For each agent $i \in \mathscr{I}$, let M_i be a non-empty set of messages. Let $M = M_1 \times M_2 \times \cdots \times M_I$ and let $g : M \to X$ be an outcome function. A pair (M, g), defined in this fashion, is called a mechanism. A strategy for agent $i \in \mathscr{I}$ is a function $m_i : \Theta_i \to M_i$ that specifies a message for each type of agent i.

Notational remark. We often need to specify the strategies of all agents but i. We do so by using m_{-i}, which stands for the profile of strategies $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_I)$.

Dominant Strategy Equilibrium

Definition

The strategy profile $m^* = (m_1^*, \ldots, m_I^*) \in M$ is a dominant strategy equilibrium of the mechanism (M, g) if, for each $\theta = (\theta_1, \ldots, \theta_I) \in \Theta_1 \times \cdots \times \Theta_I$, each $i \in \mathscr{I}$, each $\hat{m}_i \in M_i$, and each $m_{-i} : \Theta_{-i} \to M_{-i}$, we have

$$g(m_i^*(\theta_i), m_{-i}(\theta_{-i}))\ R^i(\theta_i)\ g(\hat{m}_i, m_{-i}(\theta_{-i}))$$

A dominant strategy equilibrium is a profile of strategies such that each strategy in the profile is a weakly dominant strategy for every type of the agent who plays it.

Dominant Strategy Implementation: SCCs

Definition

A social choice correspondence $F : \Theta \to\to X$ is implemented in dominant strategies by the mechanism (M, g) if (i) (M, g) has at least one dominant strategy equilibrium, (ii) for each dominant strategy equilibrium m^* and each $\theta \in \Theta$ we have $g(m^*(\theta)) \in F(\theta)$, and (iii) for each $\theta \in \Theta$ and each $x \in F(\theta)$, there exists a dominant strategy profile $m^{*\prime}$ such that $g(m^{*\prime}(\theta)) = x$.

Dominant Strategy Implementation: SCFs

Definition

A social choice function $f : \Theta \to X$ is implemented in dominant strategies by the mechanism (M, g) if (i) (M, g) has at least one dominant strategy equilibrium m^* and (ii) for each $\theta \in \Theta$, $g(m^*(\theta)) = f(\theta)$.

Direct Mechanisms

Definition

A direct mechanism is a mechanism (M, g) such that $M = \Theta$.

In a direct mechanism, each agent can send as a message one of her types and nothing else.

Truthful Dominant Strategy Implementation: SCCs

Definition

A social choice correspondence $F : \Theta \to\to X$ is truthfully implemented in dominant strategies if there exists a direct mechanism (Θ, g) which has at least one dominant strategy equilibrium m^* such that for each $\theta \in \Theta$, $g(m^*(\theta)) \in F(\theta)$.

This requires that one dominant strategy equilibrium of the mechanism leads to an outcome in the desired set, for each environment. However, it is possible that there is some other equilibrium of the mechanism that leads to an outcome outside the desired set, for some environment (this may be a Nash equilibrium of the mechanism).

Truthful Dominant Strategy Implementation: SCFs

Definition

A social choice function $f : \Theta \to X$ is truthfully implemented in dominant strategies if there exists a direct mechanism (Θ, g) which has at least one dominant strategy equilibrium m^* such that for each $\theta \in \Theta$, $g(m^*(\theta)) = f(\theta)$.

Dominant Strategy Incentive Compatibility

Definition

The social choice function $f : \Theta \to X$ is dominant strategy incentive compatible if the direct mechanism (Θ, f) truthfully implements f in dominant strategies.

A social choice function is dominant strategy incentive compatible if and only if it is strategy-proof.

Revelation Principle for Dominant Strategies

This fundamental result tells us a lot about the potential and limitations of implementation via dominant strategies.

Theorem (Revelation Principle for Dominant Strategies)

Assume that the mechanism (M, g) implements the social choice correspondence $F : \Theta \rightarrow\rightarrow X$ in dominant strategies. Then there exists a social choice function $f' : \Theta \rightarrow X$ that is dominant strategy incentive compatible.

The proof follows directly from the relevant definitions.

If F is in fact a social choice function f (it assigns to every θ a single outcome in X), then $f' = f$ in the Revelation Principle.

Implications of the Revelation Principle

1. The search for mechanism to implement an SCC in dominant strategies can be narrowed down to a search in the space of direct mechanisms for a strategy-proof direct mechanism.
2. But this brings us against the negative implications of the Gibbard-Satterthwaite theorem.

Outline

Single-Peaked Preferences

- Single-peaked preferences work well in social choice theory. The reason is that, at least if there is an odd number of agents with single-peaked preferences, majority rule is well-defined.
- To define single-peaked preferences properly, we need X to be such that it can be seen as one-dimensional by the agents (and in the same way for each agent).

Definition of Single-Peakedness

Definition

Let $\succcurlyeq$ be an ordering on X that is reflexive, transitive, and total (for every $x, y \in X$ with $x \neq y$, either $x \succcurlyeq y$ or $y \succcurlyeq x$, but not both, holds). An agent's preference ordering R over X is single-peaked if there exists an $x^* \in X$ such that, for each $x', x'' \in X$, if $x^* \succcurlyeq x' \succ x''$ or $x'' \succ x' \succcurlyeq x^*$, then $x' \; P \; x''$.

- The peak is the alternative denoted by x^*.
- Only one peak can exist.
- Think of the ordering $\succcurlyeq$ as right-to-left in terms of political positions. But more examples follow.
- As you read the examples, notice how much they are of the "partial equilibrium analysis" mold.

Examples of Single-Peaked Preferences

1. Concert seating. If your ideal row for a concert is row 10, then you prefer to sit in row 12 over sitting in row 13 (or 14, or 15 …) and you prefer sitting in row 5 over sitting in row 4 or 3, and so on.
2. Joint production. A number of workers provide homogeneous labor input for a project and a fixed amount of the input is required to complete the project.
3. Indivisible Public Good. A number of condo owners consider building a swimming pool to which each one of them will have access. There is a number of possible sizes they can choose for the pool. They have agreed to equal shares of the cost. (See details in book.)

- Note how in all three examples the ordering $\geqslant$ is obvious and it is intuitive that all agents share the same ordering.

The Uniform Rule: Notation

- The uniform rule was seminally studied in this context by Yves Sprumont.
- The uniform rule cannot be defined formally without some extra notation.
- Let K be some fixed amount of a divisible commodity to be allocated among the agents. (For example, how much work to put into the project, in the second example above). This restricts X to be the interval $[0,K]$.
- Denote the peak for preference ordering R^i by $p(R^i)$.
- Let $r_i : [0,K] \to [0,K]$ the function that assigns to each $x_i \in [0,K]$ the alternative $r_i(x_i)$ on the other side of $p(R^i)$ that is indifferent to x_i if such an alternative exists; otherwise it assigns the end-point of $[0,K]$ on the other side of $p(R^i)$.

The Uniform Rule: Notation, continued

- An economy is a pair of a preference profile $\rho = (R^1, \ldots, R^I)$ for the agents and an amount K. Denote an economy by (ρ, K).
- A feasible allocation for an economy is a profile $(x_1, \ldots, x_I) \in [0, K]^I$ such that $\sum_{i=1}^{I} x_i = K$.
- A social choice function on this domain of economies assigns to each i an amount $\varphi_i(\rho, K)$.

The Pareto Axiom for the Uniform Rule

- The Pareto axiom needs to be adapted for this domain.

Definition (Pareto Axiom)

The social choice function φ satisfies the Pareto axiom if, for each (ρ, K) in the domain, $\sum_{i=1}^{I} p(R^i) \leq K$ implies that for each i we have $\varphi_i(\rho, M) \geq p(R^i)$ and $\sum_{i=1}^{I} p(R^i) \geq K$ implies that for each i we have $\varphi_i(\rho, K) \leq p(R^i)$.

The Pareto Axiom in Words

- This says that, to be Pareto efficient, the social choice function has to behave in specific ways if the sum of all the agents' peaks is less than or more than K. If the sum of the peaks is less than K, we must allocate to at least one agent an amount higher than her peak to maintain feasibility. For the allocation to be efficient, every agent must be given an amount at least as large as her peak. If not, then taking some amount away from an agent to the right of her peak and giving it to an agent to the left of her peak would be a Pareto improvement. (Same kind of story if the sum of the peaks is more than K.)

More Axioms

- Anonymity and Strategy-Proofness can easily be adapted from our previous notation to this case.
- One more axiom that has interest on this domain is no-envy.

Definition (No-Envy Axiom)

The social choice function φ is envy-free if for every economy (ρ, K) and every pair i, j of agents, $\varphi_i(\rho, K)\ R^i\ \varphi_j(\rho, K)$.

The Uniform Rule

Definition

For any preference profile $\rho = (R^1, \ldots, R^I)$, let $\lambda(\rho)$ be implicitly defined by $\sum_{i=1}^{I} \min\{p(R^i), \lambda(\rho)\} = K$ and let $\mu(\rho)$ be implicitly defined by $\sum_{i=1}^{I} \max\{p(R^i), \mu(\rho)\} = K$. Then the uniform rule is the social choice function $\varphi^U : \mathscr{R}^I \times [0, K] \to [0, K]^I$ given by, for each $i \in \mathscr{I}$,

$$\varphi_i^U(\rho, K) = \begin{cases} \min\{p(R^i), \lambda(\rho)\} & \text{if } \sum_{i=1}^{I} p(R^i) \geq K, \\ \max\{p(R^i), \mu(\rho)\} & \text{if } \sum_{i=1}^{I} p(R^i) < K. \end{cases}$$

The Uniform Rule: Example

(From Gaertner, A Primer in Social Choice Theory, Oxford 2006, page 176.)

- Let $I = 3$. Let ρ be such that $p(R^1) = 2$, $p(R^2) = 4$, $p(R^3) = 6$.
- $K = 1.5$; $\lambda(\rho) = 0.5$; $\varphi_1(\rho, 1.5) = \varphi_2(\rho, 1.5) = \varphi_3(\rho, 1.5) = 0.5$.
- $K = 3$; $\lambda(\rho) = 1$; $\varphi_1(\rho, 3) = \varphi_2(\rho, 3) = \varphi_3(\rho, 3) = 1$.
- $K = 6$; $\lambda(\rho) = 2$; $\varphi_1(\rho, 6) = p(R^1) = 2$, $\varphi_2(\rho, 6) = \varphi_3(\rho, 6) = 2$.
- $K = 9$; $\lambda(\rho) = 3.5$; $\varphi_1(\rho, 9) = p(R^1) = 2$, $\varphi_2(\rho, 9) = \varphi_3(\rho, 9) = 3.5$.
- $K = 10$; $\lambda(\rho) = 4$; $\varphi_1(\rho, 10) = p(R^1) = 2$, $\varphi_2(\rho, 10) = p(R^2) = 4$, $\varphi_3(\rho, 10) = 4$.
- $K = 12$; $\lambda(\rho) = 6$; $\varphi_1(\rho, 12) = p(R^1) = 2$, $\varphi_2(\rho, 12) = p(R^2) = 4$, $\varphi_3(\rho, 12) = p(R^3) = 4$.
- $K = 14$; $\mu(\rho) = 4$, $\varphi_1(\rho, 14) = \varphi_2(\rho, 14) = p(R^2) = 4$, $\varphi_3(\rho, 14) = p(R^3) = 6$.

A Characterization of the Uniform Rule

Theorem

A social choice function φ satisfies the Pareto axiom and anonymity and is strategy-proof if and only if it is the uniform rule.

Another Characterization of the Uniform Rule

Theorem

A social choice function φ satisfies the Pareto axiom and is envy-free and strategy-proof if and only if it is the uniform rule.

Outline

Quasilinear Domain

Another domain of economies often studied is the partial equilibrium domain of quasilinear preferences. On this domain there is a class of mechanisms, the Groves mechanisms, with some desirable properties, including strategy-proofness.

Quasinilear Domain Setup

- Break each alternative x in two parts: $x = (d, t)$.
- The part d is a social decision and the part $t = (t_1, \ldots, t_I)$ is a list of monetary transfers to the agents. We allow each t_i to be positive, zero, or negative (if it is negative, it is a payment that agent i has to make to the central authority).
- Utility looks like this: $u_i(d, t, \theta_i) = v_i(d, \theta_i) + t_i$.

Example: Binary Public Project

- The decision set D, from which d is chosen, is $\{0,1\}$. If $d=0$ then the project is not built, if $d=1$ then the project is built.
- If the project is built, if can be of one form and size only. (Simplicity)
- Building the project costs $c>0$ monetary units. (Same as utility units, due to the quasilinearity.)
- The cost is to be split equally.
- To avoid the incentive problems of equal cost sharing, we add the possibility of transfers.
- Utility for each agent i is given by $u_i(d,t,\theta_i) = d\times\theta_i - d\times c/I + t_i$. (Multiplication signs emphasize that d here does not denote the differential operator.)

Example: Public Good with Different Provision Levels

- Extends the previous example.
- The public good level is now $y \in \mathbb{R}_+$.
- The cost of producing level y is $c(y)$.
- The set of social decisions is

$$D = \left\{ (y, z_1, \ldots, z_I) \in \mathbb{R}_+ \times \mathbb{R}^I \mid \sum_{i=1}^{I} z_i = c(y) \right\}.$$

Example: Indivisible Private Good

- Example covers single-object auctions.
- D contains I-dimensional lists $(d_1, \ldots, d_I)$ such that exactly one d_i equals 1 and all other d_j equal 0.

Decision Rules and Efficiency

Definition (Decision Rule)

A decision rule is a function $d : \Theta \to D$.

Definition (Efficient Decision Rule)

A decision rule $d : \Theta \to D$ is efficient if, for each $d' \in D$,

$$\sum_{i=1}^{I} v_i(d(\theta), \theta_i) \geq \sum_{i=1}^{I} v_i(d', \theta_i).$$

Gazebo Example

- Special case of the binary public project.
- Township must decide to build or not a gazebo.
- Set of agents contains Rich, Bob and Lori.
- Each of them has a type θ_i that measures the value s/he would get from the gazebo.
- The types are $\theta_R = 5$, $\theta_B = 15$, $\theta_L = -25$. (Lori hates looking at a gazebo.)
- Efficient decision (if the township knows the actual types) is to not build.
- Asking outright for the types invites Rich and Bob to submit inflated positive numbers for their types and Lori to submit a negative number of inflated absolute value.
- Can we provide better incentives? To be continued…

Transfer Functions

Definition

A transfer function is a function $t : \Theta \to \mathbb{R}^I$.

A transfer function is vector valued and its values can have positive (subsidy), zero, or negative (payment) components.

Social Choice Functions

Definition

A social choice function on the domain of quasilinear economies is a pair $(d(\cdot), t(\cdot))$ of a decision rule and a transfer function.

Suppose the profile $\hat{\theta}$ is announced (truthfully or not) when the true type profile is θ. Agent i gets utility

$$u_i(\hat{\theta}, d(\cdot), t(\cdot), \theta_i) = v_i(d(\hat{\theta}), \theta_i) + t_i(\hat{\theta}).$$

Feasible Transfer Functions

Definition (Feasibility)

A transfer function $t : \Theta \to \mathbb{R}^I$ is feasible if, for each $\theta \in \Theta$, $\sum_{i=1}^I t_i(\theta_i) \leq 0$.

- This says that the central authority does not run a deficit. It cannot do so because the model does not contain any source of funds that would cover the deficit. If it ran a surplus ($\sum_{i=1}^I t_i(\theta_i) < 0$), it would have to give it to an outsider (but again we have no outsiders in the model) or find a way to waste it.
- If there is a surplus, then it is wasted. So a social choice function that involves a surplus is not (first-best) efficient, because of this waste.

Balanced Transfer Functions

Definition (Balancedness)

A transfer function $t : \Theta \to \mathbb{R}^I$ is balanced if, for each $\theta \in \Theta$, $\sum_{i=1}^{I} t_i(\theta_i) = 0$.

Part I of a Fundamental Result

Theorem (Theodore Groves)

If d is an efficient decision rule and for each i there exists a function $h_i : \times_{j \in \mathcal{I}, j \neq i} \Theta_j \to \mathbb{R}$ such that, for each $\theta \in \Theta$, $t_i(\theta) = h_i(\theta_{-i}) + \sum_{j \in \mathcal{I}, j \neq i} v_j(d(\theta), \theta_j)$, then (d, t) is a dominant strategy incentive compatible (strategy-proof) social choice function.

- A mechanism with the above property is a Groves mechanism.

Part II of the Fundamental Result

Definition

A type space Θ_i is complete if $\{v_i(\cdot,\theta_i)\,|\,\theta_i \in \Theta_i\} = \{v\,|\,v : D \to \mathbb{R}\}$. In words, this says that every possible real-valued function on D results from some type θ_i.

- Note that a complete Θ_i must be uncountably infinite.

Theorem (Jerry Green and Jean-Jacques Laffont)

If $d : \Theta \to D$ is an efficient decision rule, (d,t) is strategy-proof, and the type space Θ_i of every agent $i \in I$ is complete, then for each $i \in I$ there exists a function $h_i : \times_{j\in\mathcal{I}, j\neq i}\Theta_j \to \mathbb{R}$ such that, for each $\theta \in \Theta$, $t_i(\theta) = h_i(\theta_{-i}) + \sum_{j\in\mathcal{I}, j\neq i} v_j(d(\theta),\theta_j)$.

Interlude

- We continue with an important representative of the class of Groves mechanisms, the pivotal mechanism.
- After getting its intuition, we will see how it can be used to build every other Groves mechanism.
- Finally, we go over the proof of the first part of the fundamental result on Groves mechanisms. The proof of the second part is too long for slides (but is available in the book).

The Pivotal Mechanism

- This idea is due to E. Clarke. The pivotal mechanism is often called the Clarke mechanism.
- Main idea: Internalize the effect one agent's report of her type has on everybody else.
- You only need to do this for agents whose reports may change the social decision.
- To find this effect, imagine the society with this agent removed.
- Here is the effect:

$$\sum_{j\in\mathscr{I},j\neq i} v_j(d(\theta),\theta_j) - \max_{d'\in D} \sum_{j\in\mathscr{I},j\neq i} v_j(d',\theta_j).$$

- Important: θ in the first term of this expression includes θ_i. However, θ_i is absent from the second term.

More on the Pivotal Mechanism

- The first term of the expression captures the total consumer surplus that goes to all agents except i when i's type report is considered.
- The second term captures the (first-) best that all agents except i can do by themselves.
- By definition, the second term cannot be less than the first term.
- If the first term is less than the second term we say that agent i is pivotal; he has caused harm to the other agents.
- If so, we charge i the total amount of the harm:

$$t_i(\theta) = \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta), \theta_j) - \max_{d' \in D} \sum_{j \in \mathscr{I}, j \neq i} v_j(d', \theta_j) < 0.$$

- The formula works fine for non-pivotal agents too, as the amount to charge them is simply zero.

Why the Pivotal Mechanism is a Groves Mechanism

Take h_i as follows:

$$h_i(\theta_{-i}) = -\max_{d' \in D} \sum_{j \in \mathscr{I}, j \neq i} v_j(d', \theta_j).$$

Building Groves Mechanisms from the Pivotal Mechanism

- You can build any Groves mechanism by adding an arbitrary function to the transfer function of the pivotal mechanism.
- Say you want a mechanism with a particular transfer function h_i. Then add, for each $\theta \in \Theta$, the expression $\max_{d' \in D} \sum_{j \in \mathscr{I}, j \neq i} v_j(d', \theta_j) + h_i(\theta_{-i})$ to the value of the transfer function of the pivotal mechanism at θ.

Back to the Gazebo Example

- If the township uses the pivotal mechanism, every one of the three agents will report her/his true type as her/his dominant strategy says to do so, because of the fundamental result.
- When they do so, the gazebo is not built.
- Remove Rich and Bob, one at a time, and you see that the efficient decision is still not to build. Therefore neither of them is pivotal.
- Lori is pivotal. Indeed, remove her and the efficient decision becomes to build.
- Lori should have the transfer $t_L(5, 15, -25) = 0 - (5 + 15) = -20$ at the true type values (it is negative, so she has to pay it).
- If the gazebo is built, Lori gets payoff of -25. If it is not built, her payoff is -20. It is better for her to report her true type.
- The township makes a surplus. It cannot return it to the citizens without breaking the incentive scheme.

A Variant of the Gazebo Example

- Take the example and change Lori's true type to $\theta_L = -10$.
- Under the pivotal mechanism, the gazebo is built and Bob is the pivotal agent.
- Bob's payment to the township is 5.
- Bob still has the best response to report the truth when assessed this tax.
- The township makes a surplus. It cannot return it to the citizens without breaking the incentive scheme.

Problems with Groves Mechanisms

- In the two versions of the gazebo example, the township made a surplus that it needed to waste.
- You can think of this as the price of extracting the correct behavior out of each agent as a dominant strategy.
- Is this a sign of trouble? Yes, if you regard this type of waste as trouble. But you could argue that it is simply a payment for better incentives.
- In reality, the township would be run by individuals and they would love to pocket the surplus. The incentive scheme then would not be the one we are talking about. . .
- There are other problems, beyond the balance issue. We will mention the voluntary participation problem in these notes.

Balance Problem of Groves Mechanisms

- For every Groves mechanism, it is possible to encounter the balance problem, that is, for the central authority to accumulate a surplus.
- Our two gazebo examples provide explicit proof of this.
- The book has another proof by example.

Voluntary Participation Problem of Groves Mechanisms

- Return to the first gazebo example.
- Lori has to pay 20 monetary units.
- However, the project is not built.
- Lori would have been better off not participating in the process.
- Qualification: we are imagining that if she were to not participate, the project would be abandoned and the gazebo would not be built. This implies that non-participation would yield every agent zero utility.
- Another example is in the book.

Proof of the Fundamental Result, Part I

- Suppose that d is an efficient rule, for each i there is an h_i function with the property stated in the theorem, and (d, t) is not dominant strategy incentive compatible.
- There must be an i, a θ, and a $\widehat{\theta}_i$ such that $v_i(d(\theta_{-i}, \widehat{\theta}_i), \theta_i) + t_i(\theta_{-i}, \widehat{\theta}_i) > v_i(d(\theta), \theta_i) + t_i(\theta)$.
- Use $t_i(\theta) = h_i(\theta_{-i}) + \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta), \theta_j)$ to turn this into

$$v_i(d(\theta_{-i}, \widehat{\theta}_i), \theta_i) + h_i(\theta_{-i}) + \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta_{-i}, \widehat{\theta}_i), \theta_j) >$$

$$v_i(d(\theta), \theta_i) + h_i(\theta_{-i}) + \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta), \theta_j).$$

- Note $h_i(\theta_{-i})$ is the same on both sides. (This is why it is important that h_i does not depend on θ_i.)

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- Use $t_i(\theta) = h_i(\theta_{-i}) + \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta), \theta_j)$ to turn this into

$$v_i(d(\theta_{-i}, \widehat{\theta}_i), \theta_i) + h_i(\theta_{-i}) + \sum_{j \in \mathscr{I}, j \neq i} v_j(d(\theta_{-i}, \widehat{\theta}_i), \theta_j) >$$

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- Note $h_i(\theta_{-i})$ is the same on both sides. (This is why it is important that h_i does not depend on θ_i.)

Proof of the Fundamental Result, Part I continued

- Cancel $h_i(\theta_{-i})$ from both sides.
- Consolidate the sums to get

$$\sum_{k=1}^{I} v_k(d(\theta_{-i}, \widehat{\theta}_i), \theta_k) > \sum_{k=1}^{I} v_k(d(\theta), \theta_k).$$

- Observe that this contradicts the assumed efficiency of d.
- It follows that if d is an efficient rule and for each i there is an h_i function with the property stated in the theorem, (d, t) is dominant strategy incentive compatible.

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